CONVEXITY OF RUIN PROBABILITY AND OPTIMAL DIVIDEND STRATEGIES FOR A GENERAL LÉVY PROCESS

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Abstract

We continue the recent work of Yuen and Yin (Mathematical and Computer Modelling 53 (2011): 1700-1707) to study the optimal dividends problem for a company whose cash reserves follow a general Lévy process with two-sided jumps. The objective is to find a policy which maximizes the expected discounted dividends until the time of bankruptcy. Under appropriate conditions, we obtain the convexity properties of probability of ruin and the generalized scale function. Subsequently we investigate when the dividend policy that is optimal among all admissible ones takes the form of a barrier strategy.

Keywords: barrier strategy, Bernstein function, complete monotonicity, HJB equation, Lévy process with two-sided jumps, optimal dividend problem, ruin probability, stochastic control.

1 INTRODUCTION

There are a lot of problems in insurance and finance which can be set up as optimization problems, such as the optimal investment policy, the optimal dividend payment, the optimal reinsurance problem and the optimal consumption, etc. Finding optimal dividend strategies is a classical problem in the financial and actuarial literature. The idea is that the company wants to pay some of its reserves or surpluses as dividends, and the problem is to find a dividend strategy that maximizes the expected total discounted dividends received by the shareholders until ruin. This problem goes back to De Finetti (1957), who presented his paper at the 15th International Congress of Actuaries in New York City. To make the problem tractable, he considered a discrete time risk model with step sizes ± 1 and showed that a certain barrier strategy maximizes expected discounted dividend payments. Subsequently, the problem of finding the optimal dividend strategy has become a popular topic in the actuarial literature. For the diffusion models, optimal dividend problem has been studied extensively. See, Jeanblanc and Shiryaev (1995), Asmussen and Taksar (1997), Asmussen, Hϕgaard and Taksar (2000), Gerber and Shiu (2004), Cadenillas et al. (2006), Lϕkka and Zervos (2008), Paulsen (2008), He and Liang (2009) and references therein.

For the Cramér-Lundberg risk model, the optimal dividend problem was solved by Gerber (1969), identifying so-called band strategies as the optimal ones. For exponentially distributed claim sizes this strategy simplifies to a barrier strategy. Azcue and Muler (2005) used the technique of stochastic control theory and Hamilton-Jacobi-Bellman (HJB) equation to solving this problem for Cramér-Lundberg risk model, who also included a general reinsurance strategy as a second control possibility. Albrecher and Thonhauser (2008) considered the classical risk model under a force of interest. Kulenko and Schmidli (2008) studied the optimal dividend strategies with capital injections in the classical Cramér-Lundberg risk model and moreover, Bai and Guo (2010) also considered the transaction costs and taxes on dividends. Azcue and Muler (2010) considered the optimization problem in the classical Cramér- Lundberg setting, but they allow the management the pos-

sibility of controlling the stream of dividend pay-outs and of investing part of the surplus in a Black and Scholes financial market. Avram, Palmowski and Pistorius (2007) studied the problem of maximizing the discounted dividend payout when the uncontrolled surplus of the company follows a general spectrally negative Lévy process and gives a sufficient condition involving the generator of the Lévy process for the optimal strategy to consist of a barrier strategy. Recently, Loeffen (2008) showed that barrier strategies maximize the expected discounted dividend payments until ruin also for general spectrally negative Lévy risk processes with completely monotone jump density and Kyprianou et al. (2010) relaxed this condition on the jump density to log-convex, see Yin and Wang (2009) for an alternative approach. Furthermore, Loeffen (2009a, 2009b) considered the optimal dividends problem with transaction costs and a terminal value for the spectrally negative Lévy process. Alvarez and Rakkolainen (2009) analyzed the determination of a value maximizing dividend payout policy for a broad class of cash reserve processes modeled as spectrally negative jump diffusions. Some authors considered this problem in the context of finance; See, among others, Bayraktar and Egami (2008), Belhaj (2010). Several authors consider the same problem under the framework of the dual of the Cramér-Lundberg model. We refer readers to Avanzi et al. (2007), Dai et al. (2010) and Yao et al. (2011). For more background and nice recent surveys on dividend strategies and optimal dividends problems, we refer the reader to Avanzi (2009), Albrecher and Thonhauser (2009) and Schmidli (2008).

All of those works are based on pure diffusion models or the spectrally one-sided Lévy/diffusion processes. However, few papers have dedicated to optimal dividend problem for Lévy processes with both upward and downward jumps. Inspired by the work of Avram et al. (2007), Loeffen (2008) and Kyprianou et al. (2010), Yuen and Yin (2011) considered this optimization problem for a special Lévy process with both upward and downward jumps and show that a barrier strategy forms an optimal strategy under the condition that the Lévy measure has a completely monotone density. The purpose of this paper is to examine the analogous questions in a general Lévy process framework. This optimization problem is more complex than the one we treated before, we will use the recent result on the theory of potential analysis of subordinators. The main

results of this paper are the following: If the Lévy density of negative jumps is completely monotone/log-convex, then the optimal dividend strategy is of a barrier type.

The paper is organized as follows. In Section 2 we give a rigorous mathematical formulation of the problem. In Section 3 we give a brief review on the ladder processes and potential measure for general Lévy processes. In Sections 4 and 5 we discuss the convexity of probability of ruin and the generalized scale function. In Section 6 we give the main results and their proofs. Finally, Section 7 presents an example.

2 THE MATHEMATICAL MODEL

For a rigorous mathematical formulation, we start with a real-valued Lévy process $X = \{X_t\}_{t\geq 0}$, that is, X is a stochastic process with stationary and independent increments and has right-continuous paths with left-limits, defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is generated by the process X and satisfies the usual conditions of right-continuity and completeness. Denote by P_x for the law of X when $X_0 = x$. Let E_x be the expectation associated with P_x. For short, we write P and E when $X_0 = 0$.

We assume that the Lévy triplet of X is (a, σ^2, Π), where $a, \sigma \geq 0$ are real constants, and Π is a positive measure on $(-\infty, \infty) \setminus \{0\}$ which satisfies the integrability condition

$$\int_{-\infty}^{\infty} (1 \wedge x^2)\Pi(dx) < \infty.$$

Π is called the Lévy measure and σ is the Gaussian component of X. If $\Pi(dx) = \pi(x)dx$, then we call π the Lévy density. In the sequel, we shall only consider the case that Π has a density π and the process X drifts to $+\infty$: $P(\lim_{t\to\infty} X_t = +\infty) = 1$.

The characteristic exponent of X is given by

$$\kappa(\theta) = -\frac{1}{t}\log E(e^{i\theta X_t}) = -ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{\{|x|<1\}})\Pi(dx),$$

where $\mathbf{1}_A$ is the indicator of a set A.

We define the Laplace exponent of X:

$$\Psi(\theta) = \frac{1}{t}\log Ee^{\theta X_t} = a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{-\infty}^{\infty}(e^{\theta x} - 1 - \theta x \mathbf{1}_{\{|x|<1\}})\Pi(dx). \tag{2.1}$$

Such a Lévy process is of bounded variation if and only if $\sigma = 0$ and $\int_{-1}^{1}|x|\Pi(dx) < \infty$. When $\Pi\{(0,\infty)\} = 0$, i.e. the Lévy process X with no positive jumps, is called the spectrally negative Lévy process; When $\Pi\{(-\infty,0)\} = 0$, i.e. the Lévy process X with no negative jumps, is called the spectrally positive Lévy process. A good reference for Lévy processes is Kyprianou (2006).

Set

$$\Theta = \sup\left\{\theta \in \mathbb{R} : \int_{-\infty}^{\infty} e^{\theta x}\Pi(dx) < \infty\right\}.$$

We assume throughout this paper that the Lundberg's condition holds:

$$\Theta > 0 \ \text{ and } \ \lim_{\theta\uparrow\Theta}\Psi(\theta) = +\infty. \tag{2.2}$$

Consider an insurance company or an investment company whose cash reserves (also called 'risk' or 'surplus' process) evolve according to a general Lévy process X before dividends are deducted. Let $\xi = \{L_t^\xi : t \geq 0\}$ be a dividend policy consisting of a right-continuous non-negative non-decreasing process adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of X with $L_{0-}^\xi = 0$, where L_t^ξ represents the cumulative dividends paid out up to time t. Given a control policy ξ, we assume that the controlled reserve process with initial capital $x \geq 0$ is given by $U^\xi = \{U_t^\xi : t \geq 0\}$, where

$$U_t^\xi = X_t - L_t^\xi, \tag{2.3}$$

with $X_0 = x$. Let $\tau^\xi = \{t > 0 : U_t^\xi < 0\}$ be the ruin time when the dividend payments are taking into account. Define the value function associated to a dividend policy ξ by

$$V_\xi(x) = E_x\left(\int_0^{\tau^\xi} \mathrm{e}^{-\delta t} dL_t^\xi\right),$$

where $\delta > 0$ is the discounted rate. The integral is understood pathwise in a Lebesgue-Stieltjes sense. Clearly, $V_\xi(x) = 0$ for $x < 0$. A dividend policy is called admissible if

$L_t^\xi - L_{t-}^\xi \leq U_t^\xi$ for $t < \tau^\xi$, and if $L_{\tau^\xi}^\xi - L_{\tau^\xi-}^\xi = 0$ when $\tau^\xi < \infty$. We denote Ξ the set of all admissible dividend policies. The objective is to find

$$V_*(x) = \sup_{\xi \in \Xi} V_\xi(x),$$

and an optimal policy $\xi^* \in \Xi$ such that $V_{\xi^*}(x) = V_*(x)$ for all $x \geq 0$. The function V_* is called the optimal value function.

We denote by $\xi_b = \{L_t^b : t \geq 0\}$ the barrier strategy at b and let U^b be the corresponding risk process, i.e. $U_t^b = X_t - L_t^b$. Note that $\xi_b \in \Xi$ and if $U_0^b \in [0, b]$ then the process L_t^b can be explicitly represented by

$$L_t^b = (\sup_{s \leq t} X_s - b) \vee 0.$$

Note that L_t^b is not continuous due to the positive jumps of X. If $U_0^b = x > b$, then

$$L_t^b = (x - b)\mathbf{1}_{\{t=0\}} + (\sup_{s \leq t} X_s - b) \vee 0.$$

Denote by $V_b(x)$ the dividend-value function if barrier strategy ξ_b is applied, i.e.

$$V_b(x) = E_x \left(\int_0^{\tau^{\xi_b}} e^{-\delta t} dL_t^b \right). \tag{2.4}$$

Applying Ito's formula for semimartingale we can prove that V_b is the solution of

$$\begin{cases} \Gamma V_b(x) = \delta V_b(x), & x > 0, \\ V_b(x) = 0, & x < 0, \\ V_b(0) = 0, & \sigma^2 > 0, \\ V_b(x) = x - b + V_b(b), \end{cases}$$

in which Γ is the infinitesimal generator of X:

$$\Gamma g(x) = \frac{1}{2}\sigma^2 g''(x) + a g'(x) + \int_{-\infty}^{\infty} [g(x+y) - g(x) - g'(x) y \mathbf{1}_{\{|y|<1\}}]\Pi(dy). \tag{2.5}$$

Similarly as in Yuen and Yin (2011) V_b can be expressed as

$$V_b(x) = \begin{cases} \frac{h(x)}{h'(b)}, & 0 \leq x \leq b, \\ x - b + \frac{h(b)}{h'(b)}, & x > b, \end{cases} \tag{2.6}$$

where h is the solution of

$$\begin{cases} \Gamma h(x) = \delta h(x), & x > 0, \\ h(x) = 0, & x < 0, \\ h(0) = 0, & \sigma^2 > 0, \\ h'(0) > 0. \end{cases} \tag{2.7}$$

In particular, if $\Pi\{(0,\infty)\} = 0$, $h'(0) = \frac{2}{\sigma^2}$ when $\sigma^2 > 0$ and,

$$ch(0) = 1 \text{ with } c = a - \int_{-1}^{1} x\Pi(dx) \text{ when } \sigma^2 = 0,$$

then h becomes the so called δ-scale function $W^{(\delta)}$ for spectrally negative Lévy process (cf. Avram et al. (2007)). From now on we call h the generalized scale function.

3 SOME RESULTS ON LADDER PROCESSES AND POTENTIAL MEASURE

In this section, for the readers convenience we briefly recall basic facts about ladder processes and potential measure used in this paper.

Consider the dual process $Y = \{Y_t\}_{t\geq 0}$, with $Y_0 = 0$, where $Y_t = -X_t$, $t \geq 0$. It is easy to see that that Lévy triplet of Y is $(-a, \sigma^2, \Pi_Y)$, where $\Pi_Y(dx) = \pi_X(-x)dx$. Let

$$\underline{Y_t} = \inf_{0\leq s\leq t} Y_s \text{ and } \overline{Y}_t = \sup_{0\leq s\leq t} Y_s,$$

be the processes of first infimum and the last supremum, respectively, of the the Lévy process Y. Following Klüppelberg, Kyprianou and Maller (2004) we introducing the notion of ladder processes and potential measure. Let $L = \{L_t : t \geq 0\}$ denote the local time in the time period $[0,t]$ that $\overline{Y} - Y$ spends at zero. Then $L^{-1} = \{L_t^{-1} : t \geq 0\}$ is the inverse local time such that $L_t^{-1} = \{s \geq 0 : L_s > t\}$, where we take the infimum of the empty set as ∞. Define a increasing process H by $\{H_t = Y_{L_t^{-1}} : t \geq 0\}$, i.e. the process of new maxima indexed by local time at the maximum. The processes L^{-1} and H are both defective subordinators, and we call them the ascending ladder time and ladder height process of Y. It is understood that $H_t = \infty$ when $L_t^{-1} = \infty$. Throughout the paper we

shall denoted the nondefective versions of L, L^{-1}, H by $\mathcal{L}$, $\mathcal{L}^{-1}$, $\mathcal{H}$, respectively. In fact the pair $(\mathcal{L}^{-1}, \mathcal{H})$ is a bivariate subordinator. We shall define $(\hat{L}^{-1}, \hat{H})$ the descending ladder time and the ladder height process in an analogous way (this means that $\hat{H}$ is a process which is negative valued). Because Y is drifts to $-\infty$, the decreasing ladder height process is not defective. Associated with the ascending and descending ladder processes are the bivariate renewal functions U and $\hat{U}$. The former is defined by

$$U(dx, ds) = \int_0^\infty P(H_t \in dx, L_t^{-1} \in ds)dt$$

and taking Laplace transforms shows that

$$\int_0^\infty \int_0^\infty e^{-\beta x - \alpha s} U(dx, ds) = \frac{1}{k(\alpha, \beta)}, \quad for \quad \alpha, \beta \geq 0,$$

where we denote by $k(\alpha, \beta)$ its joint Laplace exponents such that

$$k(0, \beta) = q + c\beta + \int_{(0,\infty)} (1 - e^{-\beta x}) \Pi_H(dx),$$

where $q \geq 0$ is the killing rate of H so that $q > 0$ if and only if $\lim_{t\to\infty} Y_t = -\infty$, $c \geq 0$ is the drift of H and Π_H is its jump measure. We denote the following marginal measure of $U(\cdot, \cdot)$ by

$$U(dx) := U(dx, [0, \infty)) = \int_0^\infty P(H_t \in dx)dt = \int_0^\infty e^{-qt} P(\mathcal{H}_t \in dx)dt, \ x \geq 0. \qquad (3.1)$$

The function U is called the potential/renewal measure. Similar notation will also be held for $\hat{U}$ and $\hat{k}$. We will also write Π_+ and Π_- for the restrictions of $\Pi(du)$ and $\Pi(-du)$ to $(0, \infty)$. Further, for $u > 0$, define

$$\bar{\Pi}_Y^+(u) = \Pi_Y\{(u, \infty)\}, \ \ \bar{\Pi}_Y^-(u) = \Pi_Y\{(-\infty, -u)\}, \ \ \bar{\Pi}_Y(u) = \bar{\Pi}_Y^+(u) + \bar{\Pi}_Y^-(u).$$

We now introducing the notion of a special Bernstein function and complete Bernstein function and two useful results.

Recall that a function $\phi : (0, \infty) \to (0, \infty)$ is called a Bernstein function if it admits a representation

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda x}) \mu(dx),$$

where $a \geq 0$ is the killing term, $b \geq 0$ the drift and μ a measure concentrated on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x)\mu(dx) < \infty$, called the Lévy measure.

The function ϕ is called a special Bernstein function if the function $\psi(\lambda) := \frac{\lambda}{\phi(\lambda)}$ is again a Bernstein function. Let

$$\psi(\lambda) = \tilde{a} + \tilde{b}\lambda + \int_0^\infty (1 - e^{-\lambda x})\nu(dx)$$

be the corresponding representations. It is shown in Song and Vondracěk (2006) that

$$\tilde{b} = \frac{1}{a + \mu((0, \infty))}\mathbf{1}_{\{b=0\}}, \quad \tilde{a} = \frac{1}{b + \int_0^\infty t\mu(dt)}\mathbf{1}_{\{a=0, \mu(0,\infty)<\infty\}}.$$

A possibly killed subordinator is called a special subordinator if its Laplace exponent is a special Bernstein function. It is shown in Song and Vondraček (2010) that a sufficient condition for ϕ to be special subordinator is that $\mu(x, \infty)$ is log-convex on $(0, \infty)$.

A function $\phi : (0, \infty) \to \mathbb{R}$ is called a complete Bernstein function if there exists a Bernstein function η such that

$$\phi(\lambda) = \lambda^2 \mathcal{L}\eta(\lambda), \quad \lambda > 0,$$

where $\mathcal{L}$ stands for the Laplace transform. It is known that every complete Bernstein function is a Bernstein function and that the following three conditions are equivalent:

(i) ϕ is a complete Bernstein function;

(ii) $\psi(\lambda) := \frac{\lambda}{\phi(\lambda)}$ is a complete Bernstein function;

(iii) ϕ is a Bernstein function whose Lévy measure μ is given by

$$\mu(dt) = dt \int_0^\infty e^{-st}\nu(ds),$$

where ν is a measure on $(0, \infty)$ satisfying

$$\int_0^1 \frac{1}{s}\nu(ds) < \infty, \quad \int_1^\infty \frac{1}{s^2}\nu(ds) < \infty.$$

We now give two results which are useful in potential theory and will be used in later sections of this paper. The first one is due to Kyprianou, Rivero and Song (2010); See also Song and Vondracek (2010):

Lemma 3.1. *Let H be a subordinator whose Lévy density, say $\mu(x), x > 0$, is logconvex, then the restriction of its potential measure to $(0, \infty)$ has a non-increasing and convex density. If furthermore, the drift of H is strictly positive then the density is in $C^1(0, \infty)$.*

Lemma 3.2 is due to Kingman (1967) and Hawkes (1977).

Lemma 3.2. *Suppose that H is a subordinator with the Laplace exponent ϕ and the potential measure U. Then U has a density u which is completely monotone on $(0, \infty)$ if and only if the tail of the Lévy measure is completely monotone.*

Remark 3.1. *Because the tail of the Lévy measure μ is a completely monotone function if and only if μ has a completely monotone density. Thus we have two equivalent statements: ϕ is a complete Bernstein function if and only if U has a density u which is completely monotone on $(0, \infty)$, or, equivalently, U has a density u which is completely monotone on $(0, \infty)$ if and only if μ has a completely monotone density.*

4 CONVEXITY OF PROBABILITY OF RUIN

Define the probability of ruin by

$$
\begin{aligned}
\psi(x) &= P(\text{there exists } t \geq 0 \text{ such that } x + X_t \leq 0) \\
&= P(\text{there exists } t \geq 0 \text{ such that } Y_t \geq x).
\end{aligned}
$$

It follows from Bertion and Doney (1994) that $\psi(x) = \alpha U(x, \infty)$, where $\alpha^{-1} = U(0, \infty) = \int_0^\infty P(H_t < \infty)dt$, with U given in (3.1).

For simplicity, we write the Lévy density π as

$$
\pi(x) = \begin{cases} \pi_+(x), & x > 0, \\ \pi_-(-x), & x < 0, \end{cases}
$$

where π_+, π_- are Lévy measures concentrated on $(0, \infty)$.

Recall that an infinitely differentiable function $f \in (0, \infty) \to [0, \infty)$ is called completely monotone if $(-1)^n f^{(n)}(x) \geq 0$ for all $n = 0, 1, 2, \cdots$ and all $x > 0$.

Lemma 4.1. *(Vigon (2002)) For any Lévy process X with Lévy density π the following holds:*

$$\overline{\Pi}_{\mathcal{H}}(x) = -\int_{-\infty}^{0} \hat{U}(dy)\overline{\Pi}_Y^+(x-y) = -\int_{-\infty}^{0} \hat{U}(dy)\overline{\Pi}_X^-(x-y), \quad x > 0,$$

where $Y = -X$ and $\hat{U}$ is the potential measure corresponding to $\hat{H}$.

Theorem 4.1. *Let X be any Lévy process with Lévy density π.*

(i) Suppose π_- is completely monotone on $(0, \infty)$, then the probability of ruin ψ is completely monotone on $(0, \infty)$. In particular, $\psi \in C^\infty(0, \infty)$.

(ii) Suppose π_- is logconvex on $(0, \infty)$, then

(a) ψ is convex on $(0, \infty)$;

(b) ψ' is concave on $(0, \infty)$;

(c) if X has no Gaussian component, ψ is twice continuously differentiable except at finitely or countably many points on $(0, \infty)$; if X has a Gaussian component, then $\psi \in C^2(0, \infty)$.

Proof. (i) Because π_- is completely monotone on $(0, \infty)$, By Lemma 4.1, the tail $\Pi_{\mathcal{H}}(x, \infty)$ of Lévy measure $\Pi_{\mathcal{H}}$ is a complete monotone function. It follows from Lemma 3.2, the potential measure U has a density u which is completely monotone on $(0, \infty)$. Thus the probability of ruin ψ is completely monotone on $(0, \infty)$, since $\psi(x) = \alpha U(x, \infty)$.

(ii) The logconvexity of π_- implies the logconvexity of $\overline{\Pi}_Y^+$, and hence $\overline{\Pi}_{\mathcal{H}}$ is logconvex on $(0, \infty)$ by Lemma 4.1 , since Logconvexity is preserved under mixing. It follows from Lemma 3.1 that the potential measure U has a non-increasing and convex density u, and thus $\psi' = -\alpha u$ is non-decreasing and concave on $(0, \infty)$, this proves (a) and (b). Since a convex function on $(0, \infty)$ is differentiable except at finitely or countably many points, if X has no Gaussian component, we know that ψ is twice continuously differentiable except at finitely or countably many points on $(0, \infty)$. If X has a Gaussian component, which

equivalent the drift of ascending ladder processes H is strictly positive, then it follows from Lemma 3.1 that $u \in C^1(0,\infty)$, and hence $\psi \in C^2(0,\infty)$. This ends the proof of Theorem 4.1.

5 CONVEXITY OF THE GENERALIZED SCALE FUNCTIONS

For a given generalized scale function h, define a barrier level by

$$b^* = \sup\{b \geq 0 : h'(b) \leq h'(x) \text{ for all } x \geq 0\},$$

where $h'(0)$ is understood to be the right-hand derivative at 0.

For a spectrally negative Lévy process, i.e. in the case $\Pi\{(0,\infty)\} = 0$, it was shown in Loeffen (2008) that if $\Pi(x,\infty)$ is completely monotone then ${W^{(q)}}'(x)$ is convex for $q > 0$. Note that, the latter implies that there exists an $a^* \geq 0$ such that $W^{(q)}$ is concave on $(0,a^*)$ and convex on (a^*,∞). In Kyprianou et al. (2010) it was shown that if $\Pi(x,\infty)$ has a density on $(0,\infty)$ which is non-increasing and log-convex then for each $q \geq 0$, the scale function $W^{(q)}(x)$ and its first derivative are convex beyond some finite value of x.

For the general Lévy process we get the similar results as following.

Theorem 5.1. *(i) Suppose π_- is completely monotone on $(0,\infty)$, then the derivative $h'(u)$ of generalized scale function is strictly convex on $(0,\infty)$ and $h \in C^\infty(0,\infty)$.*
(ii) Suppose π_- is logconvex on $(0,\infty)$, then the generalized scale function h and its derivative h' are strictly convex on (b^*,∞). Moreover, if X has no Gaussian component, h is twice continuously differentiable except at finitely or countably many points on $(0,\infty)$; If X has a Gaussian component, then $h \in C^2(0,\infty)$.

Proof. (i) Under the condition (2.2), for a given $\delta > 0$, since Ψ is strictly convex on $[0,\Theta)$, then there exists a unique number $\rho(\delta) \in (0,\Theta)$ such that $\Psi(\rho(\delta)) = \delta$. Let $\tilde{\psi}(u)$ is the ruin probability for the Lévy process $\tilde{X}$, where $\tilde{X}$ is a Lévy process with the Laplace exponent $\psi_{\rho(\delta)}$, which is given by $\psi_{\rho(\delta)}(\eta) = \Psi(\eta + \rho(\delta)) - \delta$. Repeating the same

argument as that in Yuen and Yin (2011) we find that the solutions of equation (2.6) are proportional to the function $[1-\tilde{\psi}(x)]e^{\rho(\delta)x}$. Moreover, $h \in C^{\infty}(0,\infty)$ and h' is strictly convex on $(0,\infty)$.

(ii) Let $\tilde{\mathcal{H}}$ ($\hat{\tilde{\mathcal{H}}}$) be the ascending (descending) ladder height process of $\tilde{Y} := -\tilde{X}$. By Lemma 4.1,

$$\overline{\Pi}_{\tilde{\mathcal{H}}}(x) = -\int_{-\infty}^{0} \hat{\tilde{U}}(dy)\overline{\Pi}^{-}_{\tilde{X}}(x-y), \quad x>0,$$

where $\hat{\tilde{U}}$ is the renewal measure corresponding to $\hat{\tilde{H}}$. Then

$$\Pi'_{\tilde{\mathcal{H}}}(x) = -e^{\rho(\delta)x}\int_{-\infty}^{0} \hat{\tilde{U}}(dy)e^{-\rho(\delta)y}\pi_{-}(y-x) \equiv e^{\rho(\delta)x}\nu_{+}(x).$$

The assumption of log-convexity of π_{-} implies that ν_{+} is log-convex, and hence $\Pi'_{\tilde{\mathcal{H}}}(x)$ is also log-convex. It follows from Lemma 1 Kyprianou and Rivero (2008) that the restriction of its potential measure to $(0,\infty)$ of a subordinator with Lévy density ν_{+} has a non-increasing and convex density, say f_{δ}. Also, the restriction of its potential measure to $(0,\infty)$ of a subordinator with Lévy density $\Pi'_{\tilde{\mathcal{H}}}(x)$ has a non-increasing and convex density, say h_{δ}. Moreover, $h_{\delta}(x) = e^{\rho(\delta)x}f_{\delta}(x)$. Thus, $\tilde{\psi}'(x) = -\tilde{\alpha}e^{\rho(\delta)x}f_{\delta}(x)$, where $\tilde{\alpha}^{-1} = \int_0^{\infty} P(\tilde{H}_t<\infty)dt$, Because $h(x) = [1-\tilde{\psi}(x)]e^{\rho(\delta)x}$, thus we have

$$h'(x) = \rho(\delta)h(x) + \alpha f_{\delta}(x), \ x>0,$$

which implies that $h'(x)$ tends to ∞ as x tends to ∞, since $\lim_{x\to\infty} h(x) = \infty$. Thus $b^{*}<\infty$. Using the same argument as that in Kyprianou, Rivero and Song (2010) we can prove that h and its derivative h' are strictly convex on (b^{*},∞). Finally, the smoothness of h is a direct consequence of Theorem 4.1.

6 MAIN RESULTS AND PROOFS

We now present the main results of this paper which concerns optimality of the barrier strategy $\xi^{b^{*}}$ for the de Finetti's dividend problem for general Lévy processes. This is a continuation of the work of Yuen and Yin (2011) in which only for a special Lévy process with both upward and downward jumps have a completely monotone density was considered.

Theorem 6.1. *Suppose ν is non-negative function on $(0,\infty)$ which is sufficiently smooth and satisfies*

$$(i)\ (\Gamma-\delta)\nu(x)\le 0,\ for\ almost\ every\ x>0;$$
$$(ii)\ \nu\ is\ concave\ on\ (0,\infty);$$
$$(iii)\ \nu'(x)\ge 1,\ \ x>0.$$

Then $\nu(x)\ge V_*(x)$.

Theorem 6.2. *Suppose that V_b (defined by (2.6)) is sufficiently smooth and satisfies*

$$(i)\quad V_b'(x)>1\ for\ all\ x\in[0,b);$$
$$(ii)\quad (\Gamma-\delta)V_b(x)\le 0,\ for\ all\ x>b.$$

Then $V_b(x)=V_*(x)$. In particular, if $(\Gamma-\delta)V_{b^*}(x)\le 0$ for all $x>b^*$, then $V_{b^*}(x)=V_*(x)$.

Theorem 6.3. *Suppose that π_- is completely monotone, then $V_{b^*}(x)=V_*(x)$. That is, the barrier strategy at b^* is an optimal strategy among all admissible strategies.*

Theorem 6.4. *Suppose that π_- is log-convex on $(0,\infty)$, then $V_{b^*}(x)=V_*(x)$. That is, the barrier strategy at b^* is an optimal strategy among all admissible strategies.*

Before proving the main results, we give two lemmas. The approach of these results are inspired by Loeffen (2008).

Lemma 6.1. *Suppose that h is sufficiently smooth and is convex in the interval (b^*,∞), then following are true:*

$$(i)\ b^*<\infty;$$
$$(ii)\ V_{b^*}'(x)\ge 1,\ for\ x\in[0,b^*]\ and\ V_{b^*}'(x)=V_x'(x)=1\ for\ x>b^*;$$
$$(iii)\ (\Gamma-\delta)V_{b^*}(x)=0,\ for\ x\in(0,b^*).$$

Proof (i) Since $\lim_{x\to\infty}h'(x)=\infty$, we have $b^*<\infty$.
(ii) For $x\in[0,b^*]$, $V_{b^*}'(x)=\frac{h'(x)}{h'(b^*)}$. It follows from the definition of b^* we have that $V_{b^*}(x)\ge$

1, for $x \in [0, b^*]$. $V'_{b^*}(x) = V'_x(x) = 1$ for $x > b^*$ follows from $V_{b^*}(x) = x - b^* + V_{b^*}(b^*)$ and, $V'_x(x) = 1$ follows from $V_x(x) = \frac{h(x)}{h'(x)}$.
(iii) $(\Gamma - \delta)V_{b^*}(x) = 0$ for $x \in (0, b^*)$ follows from $(\Gamma - \delta)h(x) = 0$ for $x \in (0, b^*)$ and (2.6).

Lemma 6.2. *Suppose that h is sufficiently smooth and is convex in the interval (b^*, ∞). Then for $x > b^*$,*

$$(i)\ V''_{b^*}(x) = 0 \le V''_x(x-),\ \ if\ \ \sigma \neq 0;$$
$$(ii)\ V'_{b^*}(u) \ge V'_x(u),\ \ u \in [0, x];$$
$$(iii)\ V_{b^*}(x) \ge V_x(x);$$
$$(iv)\ (\Gamma - \delta)V_{b^*}(x) \le 0.$$

Proof (i) If $\sigma \neq 0$, $V''_{b^*}(x) = 0$ is clear. Because $h \in C^2(0, \infty)$ and is convex in the interval (b^*, ∞), we have $V''_x(x-) := \lim_{y\uparrow x} V''_x(y) = \lim_{y\uparrow x} \frac{h''(y)}{h'(x)} \ge 0$.
(ii) For $u \in [0, b^*]$, by the definition of b^*, we have

$$V'_{b^*}(u) - V'_x(u) = \frac{h'(u)}{h'(b^*)} - \frac{h'(u)}{h'(x)} \ge 0.$$

For $u \in [b^*, x]$, by the convexity of h on (b^*, ∞), we have

$$V'_{b^*}(u) - V'_x(u) = 1 - \frac{h'(u)}{h'(x)} \ge 0.$$

(iii) Because $V_{b^*}(b^*) = \frac{h(b^*)}{h'(b^*)} \ge \frac{h(b^*)}{h'(x)} = V_x(b^*)$, and by (ii) $(V_{b^*} - V_x)$ is non-decreasing on (b^*, ∞), thus $V_{b^*}(x) \ge V_x(x)$.
(iv) For $x > b^*$, then $(\Gamma - \delta)V_x(x-) := \lim_{y\uparrow x}(\Gamma - \delta)V_x(y) = 0$. We have

$$\begin{aligned}
(\Gamma - \delta)V_{b^*}(x) &= (\Gamma - \delta)V_{b^*}(x) - (\Gamma - \delta)V_x(x-) \\
&= \frac{1}{2}\sigma^2(V''_{b^*}(x) - V''_x(x-)) + a(V'_{b^*}(x) - V'_x(x)) \\
&\quad + \int_{-\infty}^{\infty} (V_{b^*}(x+y) - V_{b^*}(x) - V'_{b^*}(x)y\mathbf{1}_{\{|y|<1\}})\pi(y)dy \\
&\quad + \int_{-\infty}^{\infty} (V_x(x+y) - V_x(x) - V'_x(x)y\mathbf{1}_{\{|y|<1\}})\pi(y)dy \\
&\quad -\delta(V_{b^*}(x) - V_x(x)) \equiv I_1 + I_2 + I_3 - I_4.
\end{aligned}$$

Lemma 6.1 (ii) and Lemma 6.2 (i) imply that $I_1 \leq 0$. Lemma 6.2 (iii) imply that $I_4 \geq 0$. Now we compute $I_2 + I_3$:

$$\begin{aligned} I_2 + I_3 &= \int_{-\infty}^{\infty} \{(V_{b^*} - V_x)(x+y) - (V_{b^*} - V_x)(x) - (V'_{b^*} - V'_x)(x) y \mathbf{1}_{\{|y|<1\}})\pi(y)dy \\ &= \int_{-\infty}^{0} \{(V_{b^*} - V_x)(x+y) - (V_{b^*} - V_x)(x) - (V'_{b^*} - V'_x)(x) y \mathbf{1}_{\{|y|<1\}})\pi(y)dy \\ &\quad + \int_{0}^{\infty} \{(V_{b^*} - V_x)(x+y) - (V_{b^*} - V_x)(x) - (V'_{b^*} - V'_x)(x) y \mathbf{1}_{\{|y|<1\}})\pi(y)dy \\ &\equiv J_1 + J_2. \end{aligned}$$

Using Lemma 6.1 (ii) and Lemma 6.2 (ii) we have $J_1 \leq 0$. For $y > 0$, we obtain

$$(V_{b^*} - V_x)(x+y) = (V_{b^*} - V_x)(x) = x - b^* + \frac{h(b^*)}{h'(b^*)} - \frac{h(x)}{h'(x)},$$

which, together with Lemma 6.2 (ii), implies that $J_2 = 0$. This proves (iv).

Proof of Theorem 6.1 The jump measure of X is denoted by

$$\mu^X = \mu^X(\omega, dt, dy) = \sum_s \mathbf{1}_{\{\triangle X_s \neq 0\}} \delta_{(s, \triangle X_s)}(dt, dy),$$

and its compensator is given by $\upsilon = \upsilon(dt, dy) = dt\Pi(dy)$. The Lévy decomposition (Protter,1992, Theorem 42) tells us:

$$\begin{aligned} X_t &= \sigma B_t + \int_{[0,t]\times\mathbb{R}} y \mathbf{1}_{\{|y|<1\}}(\mu^X - \upsilon) + at + \int_{[0,t]\times\mathbb{R}} (y - y\mathbf{1}_{\{|y|<1\}})\mu^X \\ &\equiv M_t + at + \sum_{0\leq s\leq t} \triangle X_s \mathbf{1}_{\{|y|\geq 1\}}, \end{aligned}$$

where $B = \{B_t\}_{t\geq 0}$ is a standard Brownian motion, M_t is a martingale with $M_0 = 0$.

Noting that ν is smooth enough for an application of the appropriate version of Itô's formula/the change of variables formula. In fact, if X is of bounded variation, then $\nu \in C^1(0, \infty)$ and we are allowed to use the change of variables (Theorem 31, Protter, 1992); If X has a Gaussian exponent, then $\nu \in C^2(0, \infty)$ and we are allowed to use Itô's formula (Theorem 32, Protter, 1992); If X has unbounded variation and $\sigma = 0$, then ν is twice continuously differentiable almost everywhere but is not in $C^2(0, \infty)$, we can use the Meyer-Itô's formula (Theorem 70, Protter, 1992) and product rule formula. In any cases,

for any appropriate localization sequence of stopping times $\{t_n, n \geq 1\}$, we get under P_x,

$$
\begin{aligned}
e^{-\delta(t_n\wedge\tau^\xi)}\nu(U^\xi_{t_n\wedge\tau^\xi}) - \nu(U^\xi_0) = \;& \textstyle\int_0^{t_n\wedge\tau^\xi} e^{-\delta s} dM^\xi_s + \int_0^{t_n\wedge\tau^\xi} e^{-\delta s}(\Gamma-\delta)\nu(U^\xi_{s-})ds \\
& + \textstyle\sum_{s\leq t_n\wedge\tau^\xi} \mathbf{1}_{\{\triangle L^\xi_s>0\}} e^{-\delta s}\Big\{\nu(U^\xi_{s-}+\triangle X_s-\triangle L^\xi_s) - \nu(U^\xi_{s-}+\triangle X_s) \\
& +\nu'(U^\xi_{s-}+\triangle X_s)\triangle L^\xi_s\Big\} \\
& - \textstyle\int_0^{t_n\wedge\tau^\xi} e^{-\delta s}\nu'(U^\xi_{s-})dL^\xi_s,
\end{aligned}
\tag{6.1}
$$

where

$$
\begin{aligned}
M^\xi_t \;=\;& \sum_{s\leq t} \mathbf{1}_{\{|\triangle X_s|>0\}}\Big\{\nu(U^\xi_{s-}+\triangle X_s) - \nu(U^\xi_{s-}) - \triangle X_s \nu'(U^\xi_{s-})\mathbf{1}_{\{|\triangle X_s|\leq 1\}}\Big\} \\
& - \int_0^t\int_{-\infty}^{\infty}\Big\{\nu(U^\xi_{s-}-y) - \nu(U^\xi_{s-}) + y\nu'(U^\xi_{s-})\mathbf{1}_{\{|y|\leq 1\}}\Big\}\pi(y)dyds \\
& + \int_0^t \nu'(U^\xi_{s-})dM_s
\end{aligned}
$$

is a local martingale. The concavity of ν implies that $\nu(x)-\nu(y)+(x-y)\nu'(y)\leq 0$ for any $x\leq y$. Taking expectations on both sides of (6.1) and using the conditions (i)-(iii), it follows that

$$
E_x(e^{-\delta(t_n\wedge\tau^\xi)}\nu(U^\xi_{t_n\wedge\tau^\xi})) - \nu(x) \leq -E_x\int_0^{t_n\wedge\tau^\xi} e^{-\delta s}dL^\xi_s. \tag{6.2}
$$

Letting $n\to\infty$ in (6.2) and recall that ξ is an arbitrary strategy in Ξ, we deduce that

$$
\nu(x) \geq \sup_{\xi\in\Xi} V_\xi(x) = V_*(x).
$$

This ends the proof of Theorem 6.1.

Proof of Theorem 6.2 In view of (2.6) and the conditions (i) and (ii), it follows that $(\Gamma-\delta)V_b(x)\leq 0$ for $x\in(0,\infty)\setminus\{b\}$ and $V_b'(x)\geq 1$ for $x>0$. Repeating the same argument as above leads to (6.1) to get

$$
\begin{aligned}
e^{-\delta t}V_b(U^\xi_t) - V_b(U^\xi_0) = \;& \textstyle\int_0^t e^{-\delta t}dN^\xi_s + \int_0^t e^{-\delta s}(\Gamma-\delta)V_b(U^\xi_{s-})ds \\
& + \textstyle\sum_{s\leq t}\mathbf{1}_{\{\triangle L^\xi_s>0\}}e^{-\delta s}\Big\{V_b(U^\xi_{s-}+\triangle X_s-\triangle L^\xi_s) - V_b(U^\xi_{s-}+\triangle X_s)\Big\} \\
& - \textstyle\int_{(0,t]} e^{-\delta s}V_b'(U^\xi_{s-})dL^{\xi,c}_s,
\end{aligned}
\tag{6.3}
$$

where $L_s^{\xi,c}$ is the continuous part of L_s^ξ and,

$$\begin{aligned} N_t^\xi &= \sum_{s\le t} \mathbf{1}_{\{|\triangle X_s|>0\}} \Big\{ V_b(U_{s-}^\xi + \triangle X_s) - V_b(U_{s-}^\xi) - \triangle X_s V_b'(U_{s-}^\xi)\mathbf{1}_{\{|\triangle X_s|\le 1\}} \Big\} \\ &\quad - \int_0^t \int_{-\infty}^{\infty} \Big\{ V_b(U_{s-}^\xi - y) - V_b(U_{s-}^\xi) + y V_b'(U_{s-}^\xi)\mathbf{1}_{\{|y|\le 1\}} \Big\} \pi(y)dyds \\ &\quad + \int_0^t V_b'(U_{s-}^\xi)dM_s. \end{aligned}$$

Since $P(\triangle L_s^\xi > 0, \triangle X_s < 0) = 0$ and, on $\{\triangle L_s^\xi > 0, \triangle X_s > 0\}$ we have $U_{s-}^\xi + \triangle X_s \ge b$. Consequently, $V_b'(U_{s-}^\xi + \triangle X_s) = 1$. It follows that

$$\begin{aligned} &\textstyle\sum_{s\le t} \mathbf{1}_{\{\triangle L_s^\xi>0\}} e^{-\delta s} \Big\{ V_b(U_{s-}^\xi + \triangle X_s - \triangle L_s^\xi) - V_b(U_{s-}^\xi + \triangle X_s) \Big\} \\ &\qquad = -\textstyle\sum_{s\le t} \mathbf{1}_{\{\triangle L_s^\xi>0\}} e^{-\delta s} \triangle L_s^\xi. \end{aligned}$$

We have for any appropriate localization sequence of stopping times $\{t_n, n\ge 1\}$,

$$E_x(e^{-\delta(t_n\wedge\tau^\xi)} V_b(U_{t_n\wedge\tau^\xi}^\xi)) - E_x V_b(U_0^\xi) \le -E_x \int_{[0,t_n\wedge\tau^\xi]} e^{-\delta s} dL_s^\xi. \tag{6.4}$$

Letting $n\to\infty$ in (6.4), as before, we deduce that

$$V_b(x) \ge \sup_{\xi\in\Xi} V_\xi(x) = V_*(x).$$

However,

$$V_b(x) \le \sup_{\xi\in\Xi} V_\xi(x) = V_*(x).$$

This ends the proof of Theorem 6.2.

Proof of Theorem 6.3 If π_- is completely monotone, it follows from Theorem 5.1 (i) that $h'(x)$ is strictly convex on $(0,\infty)$ and hence by (2.6) that V_{b^*} is concave on $(0,\infty)$. By Lemma 6.1 (ii) and (iii) and Lemma 6.2 (iv) we see that the conditions in Theorem 6.1 are satisfied. Thus $V_b(x) \ge V_*(x)$. Consequently, $V_b(x) = V_*(x)$ and the proof is complete.

Proof of Theorem 6.4 If π_- is logconvex on $(0,\infty)$, it follows from Theorem 5.1 (ii) that $h(x)$ is strictly convex on (b^*,∞) and, it follows from Lemma 6.2 (iv) that $(\Gamma-\delta)V_{b^*}(x)\le 0$ for all $x > b^*$. The result follows from Theorem 6.2.

7 EXAMPLE

Example 7.1. Consider a Lévy process X with the Laplace exponent Υ and the Lévy triplet (a, σ^2, Π), where

$$\Pi(dx) = \begin{cases} \Pi_+(dx), & x > 0, \\ \lambda P(dx), & x < 0, \end{cases} \tag{7.1}$$

in which Π_+ is an arbitrary Lévy measure concentrated on $(0, \infty)$, $\lambda > 0$, $P(-x)$ is a distribution on $(0, \infty)$ of phase-type. The process is called the Lévy process with arbitrary positive jumps and negative jumps of phase-type; See Asmussen *et al.* (2004). In particular, when $P'(x) = \sum_{j=1}^{n} b_j \alpha_j e^{\alpha_j x}$, $x < 0$, $\alpha_j > 0$, $b_j \geq 0$, $\sum_{j=1}^{n} b_j = 1$. The process is called the Lévy process with mixed-exponential negative jumps (see Mordecki (2004)). Some special cases are widely used in the literature of finance and optimal stopping, see Asmussen *et al.* (2004) and Mordecki (2004) for hyper-exponential model; Perry and Stadje (2000) and Kou and Wang (2003) for double exponential model. Other special cases are widely used in the ruin theory, for instance, Zhang, Hu and Li (2010), Albrecher, Gerber and Yang (2010) and Chi and Lin (2010).

Now we write out the generalized scale function for this Lévy process. From the proof to Theorem 5.1, we see that the generalized scale function are proportional to the function $[1 - \tilde{\psi}_1(x)]e^{\rho_1(\delta)x}$, where $\rho_1(\delta)$ is the positive solution of $\Upsilon(\eta) = \delta$, $\tilde{\psi}_1(u)$ is the ruin probability for the Lévy process $\tilde{X}$, which has the Laplace exponent $\kappa(\eta) := \Upsilon(\eta + \rho_1(\delta)) - \delta$. Since $E(\tilde{X}_1)$ is always positive, it follows from Asmussen *et al.* (2004) that

$$\tilde{\psi}_1(x) = \sum_j A_j e^{-R_j x},$$

where A_j's are constants and $-R_i$' are all distinct roots with negative real part of the equation $\kappa(\eta) = 0$. In particular, for the Lévy process with mixed-exponential negative jumps we have (see Mordecki (2004)):

$$\tilde{\psi}_1(x) = \sum_{j=1}^{n+1} A_j e^{-R_j x},$$

where $-R_i$'s $(0 < R_1 < \alpha_1 < R_2 < \cdots < \alpha_n < R_{n+1})$ are the $n+1$ negative roots of

$\kappa(\eta) = 0$ and the constants $A_1, \cdots, A_{n+1}$ are given by

$$A_j = \frac{\prod_{k=1}^{n}(1 - R_j/\alpha_k)}{\prod_{k=1,k\neq j}^{n+1}(1 - R_j/R_k)}, \; j = 1, 2, \cdots, n+1.$$

It is easy to see that all A_j's are positive, so that $\tilde{\psi}_1(x)$ is completely monotone on $(0, \infty)$.

Note that in this case the Lévy density of negative jumps in (7.1) can be written as

$$\pi_-(-x) = P'(x) = \sum_{j=1}^{n} b_j \alpha_j e^{\alpha_j x}, x < 0.$$

That is to say π_- is completely monotone on $(0, \infty)$. A direct application of Theorem 6.3 gives that the barrier strategy at b^* is an optimal strategy among all admissible strategies.

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